Multiple orthogonal polynomials associated with an exponential cubic weight

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Abstract

We consider multiple orthogonal polynomials associated with the exponential cubic weight e^{-x^3} over two contours in the complex plane. We study the basic properties of these polynomials, including the Rodrigues formula and nearest-neighbor recurrence relations. It turns out that the recurrence coefficients are related to a discrete Painlevé equation. The asymptotics of the recurrence coefficients, the ratio of the diagonal multiple orthogonal polynomials and the (scaled) zeros of these polynomials are also investigated.

Keywords: multiple orthogonal polynomials, exponential cubic weight, Rodrigues formula, nearest-neighbor recurrence relations, string equations, discrete Painlevé equation, zeros, asymptotics

1 Introduction and statement of the results

1.1 Orthogonal polynomials associated with an exponential cubic weight

A sequence of non-constant monic polynomials $\{p_n\}$ with $\deg p_n \leq n$ is said to be orthogonal with respect to the exponential cubic weight e^{-x^3} if

$$\int_\Gamma p_n(x)x^k e^{-x^3}dx = 0, \qquad k = 0, 1, \ldots, n-1, \tag{1.1}$$

where the contour Γ is chosen such that the above integral converges. These polynomials satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n^2 p_{n-1}(x), \tag{1.2}$$

where

$$\beta_n = \frac{\int_\Gamma xp_n^2(x)e^{-x^3}dx}{\int_\Gamma p_n^2(x)e^{-x^3}dx}, \qquad \gamma_n^2 = \frac{\int_\Gamma xp_n(x)p_{n-1}(x)e^{-x^3}dx}{\int_\Gamma p_{n-1}^2(x)e^{-x^3}dx}, \tag{1.3}$$

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and the initial condition is taken to be $\gamma_0^2 p_{-1} = 0$. It is shown by A. Magnus [19] that the recurrence coefficients β_n and γ_n^2 satisfy the "string" equations

$$\gamma_{n+1}^2 + \beta_n^2 + \gamma_n^2 = 0, \tag{1.4}$$

$$3\gamma_n^2(\beta_{n-1} + \beta_n) = n. \tag{1.5}$$

For the convenience of the reader, we derive the string equations using ladder operators for orthogonal polynomials in the Appendix. Some variants of orthogonal polynomials associated with the exponential cubic weight have recently been studied in the context of numerical analysis [8] and random matrix theory [4].

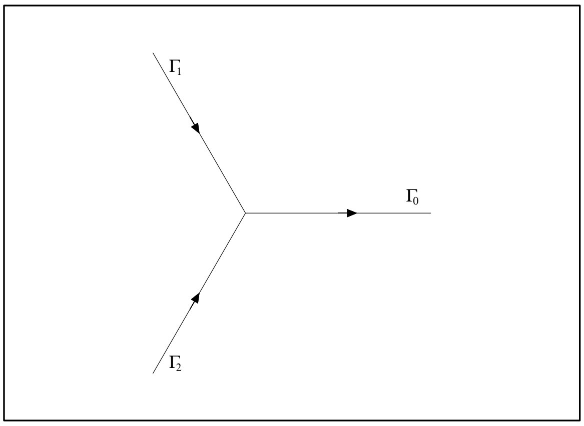


Figure 1: The three rays $\Gamma_0, \Gamma_1, \Gamma_2$

For our purpose, we are concerned with the polynomials for specific contours Γ. Consider the three rays (see Figure 1)

$$\Gamma_k = \{z \in \mathbb{C} : \arg z = \omega^k\}, \qquad k = 0, 1, 2, \tag{1.6}$$

where $\omega = e^{2\pi i/3}$ is the primitive third root of unity and the orientations are all taken from left to right. Clearly, the integral (1.1) is well-defined for each Γ_k. We shall denote by $p_n^{(1)}$ the polynomials satisfying (1.1) with $\Gamma = \Gamma_0 \cup \Gamma_1$. The corresponding recurrence coefficients will be accordingly denoted by $\beta_n^{(1)}$ and $(\gamma_n^{(1)})^2$. Hence, we have

$$\int_{\Gamma_0 \cup \Gamma_1} p_n^{(1)}(x) x^k e^{-x^3} dx = 0, \qquad k = 0, 1, \ldots, n-1, \tag{1.7}$$

and

$$x p_n^{(1)}(x) = p_{n+1}^{(1)}(x) + \beta_n^{(1)} p_n^{(1)}(x) + (\gamma_n^{(1)})^2 p_{n-1}^{(1)}(x). \tag{1.8}$$

From (1.3), it is readily seen that

$$\beta_0^{(1)} = \frac{\int_{\Gamma_0 \cup \Gamma_1} x e^{-x^3} dx}{\int_{\Gamma_0 \cup \Gamma_1} e^{-x^3} dx} = \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3}. \tag{1.9}$$

Thus, one can determine $(\beta_n^{(1)}, (\gamma_n^{(1)})^2)$ recursively from the string equations (1.4)–(1.5) with initial condition $\gamma_0^{(1)} = 0$ and $\beta_0^{(1)} = \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3}$.

In a similar manner, we let $p_n^{(2)}$ be the polynomials satisfying (1.1) with $\Gamma = \Gamma_0 \cup \Gamma_2$, and denote by $\beta_n^{(2)}$ and $(\gamma_n^{(2)})^2$ the corresponding recurrence coefficients. To this end, one has

$$\beta_0^{(2)} = \frac{\int_{\Gamma_0 \cup \Gamma_2} x e^{-x^3} dx}{\int_{\Gamma_0 \cup \Gamma_2} e^{-x^3} dx} = \overline{\beta_0^{(1)}} = \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{-\pi i/3}. \tag{1.10}$$

For the recurrence coefficients $\beta_n^{(i)}$ and $(\gamma_n^{(i)})^2$, $i = 1, 2$, the following proposition holds.

Proposition 1.1. *There exist two real sequences a_n and b_n, $n \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ such that*

$$\beta_n^{(1)} = b_n e^{\pi i/3}, \qquad (\gamma_n^{(1)})^2 = a_n e^{-\pi i/3}, \tag{1.11}$$

and a_n, b_n satisfy the coupled difference relations

$$a_n + a_{n+1} = b_n^2, \tag{1.12}$$

$$3a_n(b_n + b_{n-1}) = n, \tag{1.13}$$

with initial conditions

$$a_0 = 0, \qquad b_0 = \frac{\Gamma(2/3)}{\Gamma(1/3)}. \tag{1.14}$$

Similarly, we have

$$\beta_n^{(2)} = b_n e^{-\pi i/3}, \qquad (\gamma_n^{(2)})^2 = a_n e^{\pi i/3}, \tag{1.15}$$

with the same sequences a_n and b_n.

From (1.13), one can easily eliminate a_n in (1.12) and obtain

$$\frac{n}{b_{n-1} + b_n} + \frac{n+1}{b_n + b_{n+1}} = 3b_n^2. \tag{1.16}$$

This difference equation belongs to A_1^c-type equation on the list of discrete Painlevé equations by Grammaticos and Ramani [16, 17], which has a connection with the second Painlevé equation. It is also an alternative discrete Painlevé I equation in Clarkson's list [26, Appendix A.4], see also [11], [20]. We give a short derivation of the string equations (1.4)–(1.5) in the Appendix, where we also deal with the more general weight e^{-x^3+tx}.

1.2 Multiple orthogonal polynomials with an exponential cubic weight

Multiple orthogonal polynomials are polynomials of one variable which are defined by orthogonality relations with respect to r different measures $\mu_1, \mu_2, \ldots, \mu_r$, where $r \geq 1$ is a positive integer. As a generalization of orthogonal polynomials, multiple orthogonal polynomials originated from Hermite-Padé approximation in the context of irrationality and transcendence proofs in number theory. They were further developed in approximation theory, we refer to Aptekarev et al. [1, 2], Coussement and Van Assche [28], Nikishin and Sorokin [21, Chapter 4, §3], and Ismail [18, Chapter 23] for more information.

We take $r = 2$ and for $(k, l) \in \mathbb{N}^2$, we are interested in the monic polynomials $P_{k,l}$ of degree $k + l$ which satisfy the orthogonality conditions

$$\int_{\Gamma_0 \cup \Gamma_1} x^i P_{k,l}(x) e^{-x^3} dx = 0, \qquad i = 0, 1, \ldots, k-1, \tag{1.17}$$

$$\int_{\Gamma_0 \cup \Gamma_2} x^i P_{k,l}(x) e^{-x^3} dx = 0, \qquad i = 0, 1, \ldots, l-1. \tag{1.18}$$

We call $P_{k,l}$ the (type II) multiple orthogonal polynomial for the exponential cubic weight. If one of k and l is equal to zero, then $P_{k,l}$ reduce to the usual orthogonal polynomials with respect to the exponential cubic weight e^{-x^3}, i.e.,

$$P_{k,0}(x) = p_k^{(1)}(x), \qquad P_{0,k}(x) = p_k^{(2)}(x), \tag{1.19}$$

where $p_k^{(i)}$, $i = 1, 2$ are defined in Section 1.1. It is the aim of this paper to derive some basic properties of $P_{k,l}$. Our main results are

Theorem 1.2 (Rodrigues formula)**.** *Let $n, m \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, then*

$$e^{-x^3} P_{n,n+m}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{0,m}(x) \right), \tag{1.20}$$

$$e^{-x^3} P_{n+m,n}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{m,0}(x) \right). \tag{1.21}$$

where $P_{0,m}(x)$ and $P_{m,0}(x)$ are given in (1.19)*.*

The polynomials $P_{n,n}(x)$ were already mentioned by Pólya and Szegö in their problem book [24, Part V, Chapter 1, Problem 59] and Pólya investigated their zeros in [23, Satz IV]. They are also a special case of polynomials introduced by Gould and Hopper [15] and were investigated, among others, by Dominici [9] and Paris [22]. Their multiple orthogonality (or d-orthogonality, if one only considers the diagonal polynomials) was already noted earlier, see e.g., [3] and references there. In this paper we are investigating the full range of polynomials $P_{n,m}(x)$ and not only the diagonal polynomials, but we obtain ratio asymptotics and the distribution of the zeros for the diagonal polynomials in Section 4. For asymptotic approximations and an asymptotic expansion of $P_{n,n}(x)$ we refer to [9] and [22].

Multiple orthogonal polynomials satisfy a system of nearest-neighbor recurrence relations [18, Theorem 23.7]. For $P_{k,l}$ defined in (1.17)–(1.18) we can represent the recurrence coefficients explicitly in terms of the sequences a_n and b_n in Proposition 1.1, as stated in the following theorem.

Theorem 1.3 (the nearest-neighbor recurrence relations)**.** *Let $n, m \in \mathbb{N}$, then*

$$\begin{aligned} xP_{n,n+m}(x) = P_{n+1,n+m}(x) &+ c_{n,n+m} P_{n,n+m}(x) \\ &+ a_{n,n+m} P_{n-1,n+m}(x) + b_{n,n+m} P_{n,n+m-1}(x), \end{aligned} \tag{1.22}$$

$$\begin{aligned} xP_{n,n+m}(x) = P_{n,n+m+1}(x) &+ d_{n,n+m} P_{n,n+m}(x) \\ &+ a_{n,n+m} P_{n-1,n+m}(x) + b_{n,n+m} P_{n,n+m-1}(x), \end{aligned} \tag{1.23}$$

where

$$c_{n,n+m} = \begin{cases} \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3}, & m = 0, \\ -b_{m-1} e^{-\pi i/3}, & m > 0, \end{cases} \tag{1.24}$$

$$d_{n,n+m} = b_m e^{-\pi i/3}, \tag{1.25}$$

$$a_{n,n+m} = \begin{cases} -\frac{n}{3\sqrt{3}} \frac{\Gamma(1/3)}{\Gamma(2/3)} i, & m = 0, \\ -\frac{n a_m}{m} e^{\pi i/3}, & m > 0, \end{cases} \tag{1.26}$$

$$b_{n,n+m} = \begin{cases} \frac{n}{3\sqrt{3}} \frac{\Gamma(1/3)}{\Gamma(2/3)} i, & m = 0, \\ \frac{(n+m) a_m}{m} e^{\pi i/3}, & m > 0. \end{cases} \tag{1.27}$$

Similarly,

$$\begin{aligned} xP_{n+m,n}(x) = P_{n+m+1,n}(x) + c_{n+m,n} P_{n+m,n}(x) \\ + a_{n+m,n} P_{n+m-1,n}(x) + b_{n+m,n} P_{n+m,n-1}(x), \end{aligned} \tag{1.28}$$

$$\begin{aligned} xP_{n+m,n}(x) = P_{n+m,n+1}(x) + d_{n+m,n} P_{n+m,n}(x) \\ + a_{n+m,n} P_{n+m-1,n}(x) + b_{n+m,n} P_{n+m,n-1}(x), \end{aligned} \tag{1.29}$$

where

$$c_{n+m,n} = b_m e^{\pi i/3}, \tag{1.30}$$

$$d_{n+m,n} = \begin{cases} \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{-\pi i/3}, & m = 0, \\ -b_{m-1} e^{\pi i/3}, & m > 0, \end{cases} \tag{1.31}$$

$$a_{n+m,n} = \begin{cases} -\frac{n}{3\sqrt{3}} \frac{\Gamma(1/3)}{\Gamma(2/3)} i, & m = 0, \\ \frac{(n+m) a_m}{m} e^{-\pi i/3}, & m > 0, \end{cases} \tag{1.32}$$

$$b_{n+m,n} = \begin{cases} \frac{n}{3\sqrt{3}} \frac{\Gamma(1/3)}{\Gamma(2/3)} i, & m = 0, \\ -\frac{n a_m}{m} e^{-\pi i/3}, & m > 0. \end{cases} \tag{1.33}$$

Here, a_n and b_n are the two real sequences generated from (1.12)–(1.14)*.*

It is also easy to check that the recurrence coefficients derived in Theorem 1.3 satisfy the partial difference equations obtained in [27, Theorem 3.2].

The rest of this paper is organized as follows. Theorems 1.2 and 1.3 will be proved in Section 2. The string equation (1.4) plays a particular role in the derivation of the coefficients in the nearest-neighbor recurrence relations. We then perform a numerical study of the coefficients a_n, b_n in Section 3. The study suggests that a_{n+1} and b_n, $n \in \mathbb{N}$ are all strictly positive, and the limits of $a_n/n^{2/3}$ and $b_n/n^{1/3}$ exist as $n \to \infty$, and we can identify these limits explicitly. Section 4 deals with the zeros of $P_{k,l}$. We will give precise location and interlacing results for the zeros of the diagonal multiple orthogonal polynomials $P_{n,n}$ and asymptotic results for the ratio of diagonal multiple orthogonal polynomials. The latter allows us to find the asymptotic distribution of the scaled zeros for these diagonal multiple orthogonal polynomials. The zeros of $P_{k,l}$, with $k \neq l$, have a more interesting structure, which depends on the limit of the ratio k/l. We investigate these zeros numerically and end this paper with some conclusions and outlook.

2 Proofs

2.1 Proof of Proposition 1.1

This proposition can be proved by induction on the index n. When $n = 0$, the relation (1.11) is obvious, which also gives the initial conditions (1.14). Suppose we have

$$\beta_k^{(1)} = b_k e^{\pi i/3}, \qquad (\gamma_k^{(1)})^2 = a_k e^{-\pi i/3}, \tag{2.1}$$

and $(a_k, b_k) \in \mathbb{R}^2$ for $k \le n$. From (1.4), it follows that

$$\begin{aligned}(\gamma_{n+1}^{(1)})^2 &= -((\gamma_n^{(1)})^2 + (\beta_n^{(1)})^2) \\ &= -a_n e^{-\pi i/3} - b_n^2 e^{2\pi i/3} = (b_n^2 - a_n) e^{-\pi i/3},\end{aligned} \tag{2.2}$$

thus,

$$a_{n+1} = b_n^2 - a_n \in \mathbb{R}. \tag{2.3}$$

On the other hand, the equation (1.5) implies that

$$\beta_{n+1}^{(1)} = \frac{n+1}{3(\gamma_{n+1}^{(1)})^2} - \beta_n^{(1)} = \left(\frac{n+1}{3a_{n+1}} - b_n\right) e^{\pi i/3}, \tag{2.4}$$

thus

$$b_{n+1} = \frac{n+1}{3a_{n+1}} - b_n \in \mathbb{R}. \tag{2.5}$$

The coupled difference equations (1.12)–(1.13) are immediate from (2.3) and (2.5).

The claim for $\beta_n^{(2)}$ and $(\gamma_n^{(2)})^2$ can be proved similarly, we omit the details here.

2.2 Proof of Theorem 1.2

We shall only prove (1.20) since the proof of (1.21) is similar.

We first show that $P_{n,n+m}$ defined in (1.20) is a monic polynomial of degree $2n + m$. Observe that

$$\frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(e^{-x^3} P_{0,m}\right) = \frac{(-1)^n}{3^n} \left(\frac{d^{n-1}}{dx^{n-1}} \left(e^{-x^3} P_{0,m}\right)\right)' = -\frac{1}{3}(e^{-x^3} P_{n-1,n+m-1}(x))',$$

we then obtain from (1.20) the following difference-differential equation for $P_{n,n+m}$:

$$P_{n,n+m}(x) = x^2 P_{n-1,n+m-1}(x) - \frac{1}{3} P'_{n-1,n+m-1}(x). \tag{2.6}$$

We can now use induction on n. Clearly $P_{0,m} = p_m^{(2)}$ is a monic polynomial of degree m. Suppose that $P_{n-1,m+n-1}$ is a monic polynomial of degree $2n + m - 2$, then (2.6) implies that $P_{n,n+m}$ is a monic polynomial of degree $2n + m$.

Next, we show that $P_{n,n+m}$ satisfies the orthogonality conditions (1.17)–(1.18). With Γ_0 defined in (1.6), it follows from (1.20) and integration by parts k times that

$$\begin{aligned}\int_{\Gamma_0} x^k P_{n,n+m}(x) e^{-x^3}\, dx &= \frac{(-1)^n}{3^n} \int_{\Gamma_0} x^k \frac{d^n}{dx^n} \left(e^{-x^3} P_{0,m}(x)\right) dx \\ &= -\frac{(-1)^{n+k} k!}{3^n} \frac{d^{n-k-1}}{dx^{n-k-1}} \left(e^{-x^3} P_{0,m}(x)\right)\Big|_{x=0} \\ &= \frac{k!}{3^{k+1}} P_{n-k-1,n+m-k-1}(0),\end{aligned} \tag{2.7}$$

for $k = 0, 1, \dots, n-1$. Similarly, it is easily seen that

$$\int_{\Gamma_1} x^k P_{n,n+m}(x) e^{-x^3}\, dx = \int_{\Gamma_2} x^k P_{n,n+m}(x) e^{-x^3}\, dx = -\frac{k!}{3^{k+1}} P_{n-k-1,n+m-k-1}(0). \tag{2.8}$$

Combining (2.7) and (2.8) gives

$$\int_{\Gamma_0\cup\Gamma_1} x^k P_{n,n+m}(x) e^{-x^3}\, dx = 0, \qquad k = 0, 1, \dots, n-1,$$
$$\int_{\Gamma_0\cup\Gamma_2} x^k P_{n,n+m}(x) e^{-x^3}\, dx = 0, \qquad k = 0, 1, \dots, n-1.$$

We still need m more orthogonality conditions to complete (1.18), but these follow from

$$\begin{aligned}\int_{\Gamma_0\cup\Gamma_2} x^{n+k} P_{n,n+m}(x) e^{-x^3}\, dx &= \frac{(-1)^n}{3^n} \int_{\Gamma_0\cup\Gamma_2} x^{n+k} \frac{d^n}{dx^n}\left(e^{-x^3} P_{0,m}(x)\right) dx \\ &= \frac{(n+k)!}{k!3^n} \int_{\Gamma_0\cup\Gamma_2} x^k e^{-x^3} P_{0,m}(x)\, dx = 0\end{aligned}$$

for $k = 0, 1, \dots, m-1$, where we used the fact that $P_{0,m} = p_m^{(2)}$ is the orthogonal polynomial for the cubic exponential weight on $\Gamma_0 \cup \Gamma_2$.

2.3 Proof of Theorem 1.3

We will present the proof of (1.22)–(1.27), the remaining part of the theorem can be proved in a similar manner.

Let us denote the coefficients of x^{k+l-1} and x^{k+l-2} in $P_{k,l}$ by $\delta_{k,l}$ and $\varepsilon_{k,l}$, respectively, i.e.,

$$P_{k,l}(x) = x^{k+l} + \delta_{k,l} x^{k+l-1} + \varepsilon_{k,l} x^{k+l-2} + \cdots. \tag{2.9}$$

Substituting the above formula into (2.6) and comparing the coefficients of x^{2n+m-1} and x^{2n+m-2} on both sides leads to

$$\delta_{n,n+m} = \delta_{n-1,n+m-1}, \qquad \varepsilon_{n,n+m} = \varepsilon_{n-1,n+m-1},$$

thus,

$$\delta_{n,n+m} = \delta_{0,m}, \qquad \varepsilon_{n,n+m} = \varepsilon_{0,m}, \tag{2.10}$$

for $m \in \mathbb{N}$. Similarly, we have

$$P_{n+m,n}(x) = x^2 P_{n+m-1,n-1}(x) - \frac{1}{3} P'_{n+m-1,n-1}(x), \tag{2.11}$$

which implies

$$\delta_{n+m,n} = \delta_{m,0}, \qquad \varepsilon_{n+m,n} = \varepsilon_{m,0}. \tag{2.12}$$

If we insert (2.9) into (1.22)–(1.23), then the coefficients of second leading term x^{2n+m} give

$$c_{n,n+m} = \delta_{n,n+m} - \delta_{n+1,n+m} = \begin{cases} \delta_{0,0} - \delta_{1,0}, & m = 0, \\ \delta_{0,m} - \delta_{0,m-1}, & m > 1, \end{cases} \tag{2.13}$$

$$d_{n,n+m} = \delta_{n,n+m} - \delta_{n,n+m+1} = \delta_{0,m} - \delta_{0,m+1}, \tag{2.14}$$

where we have also made use of the first equalities in (2.10) and (2.12). On account of the facts that

$$xP_{m,0}(x) = P_{m+1,0}(x) + \beta_m^{(1)} P_{m,0}(x) + (\gamma_m^{(1)})^2 P_{m-1,0}(x), \tag{2.15}$$

$$xP_{0,m}(x) = P_{0,m+1}(x) + \beta_m^{(2)} P_{0,m}(x) + (\gamma_m^{(2)})^2 P_{0,m-1}(x), \tag{2.16}$$

(see (1.19) and (1.8)), it is immediate that

$$\delta_{m,0} = \delta_{m+1,0} + \beta_m^{(1)} = \delta_{m+1,0} + b_m e^{\pi i/3}, \tag{2.17}$$

$$\delta_{0,m} = \delta_{0,m+1} + \beta_m^{(2)} = \delta_{0,m+1} + b_m e^{-\pi i/3}, \tag{2.18}$$

in view of (1.11) and (1.15). The values for $c_{n,n+m}$, $d_{n,n+m}$ in (1.24)–(1.25) then follow from combining (2.13), (2.14) and (2.17)–(2.18).

We now establish the equalities (1.26)–(1.27) for $a_{n,n+m}$ and $b_{n,n+m}$. Multiplying both sides of (1.22) by $x^{n+m-1}e^{-x^3}$ and integrating the equality over $\Gamma_0 \cup \Gamma_2$, the orthogonality condition (1.18) implies

$$b_{n,n+m} = \frac{\displaystyle\int_{\Gamma_0\cup\Gamma_2} x^{n+m} P_{n,n+m}(x) e^{-x^3} dx}{\displaystyle\int_{\Gamma_0\cup\Gamma_2} x^{n+m-1} P_{n,n+m-1}(x) e^{-x^3} dx}. \tag{2.19}$$

By (1.20), (1.21) and integrating by parts, we find that

$$\begin{aligned}\int_{\Gamma_0\cup\Gamma_2} x^{n+m} P_{n,n+m}(x) e^{-x^3} dx &= \frac{(-1)^n}{3^n} \int_{\Gamma_0\cup\Gamma_2} x^{n+m} \frac{d^n}{dx^n}\left(e^{-x^3} P_{0,m}(x)\right) dx \\ &= \frac{(n+m)!}{3^n m!} \int_{\Gamma_0\cup\Gamma_2} x^m P_{0,m}(x) e^{-x^3} dx \\ &= \frac{(n+m)!}{3^n m!} \int_{\Gamma_0\cup\Gamma_2} P_{0,m}^2(x) e^{-x^3} dx,\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\int_{\Gamma_0\cup\Gamma_2} x^{n-1} P_{n,n-1}(x) e^{-x^3} dx &= \frac{(-1)^{n-1}}{3^{n-1}} \int_{\Gamma_0\cup\Gamma_2} x^{n-1} \frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^3} P_{1,0}(x)\right) dx \\ &= \frac{(n-1)!}{3^{n-1}} \int_{\Gamma_0\cup\Gamma_2} P_{1,0}(x) e^{-x^3} dx.\end{aligned} \tag{2.21}$$

Hence, we can simplify (2.19) as

$$b_{n,n+m} = \begin{cases} \dfrac{n}{3} \dfrac{\int_{\Gamma_0\cup\Gamma_2} e^{-x^3} dx}{\int_{\Gamma_0\cup\Gamma_2} P_{1,0}(x) e^{-x^3} dx}, & m = 0, \\[2ex] \dfrac{n+m}{m} \dfrac{\int_{\Gamma_0\cup\Gamma_2} P_{0,m}^2(x) e^{-x^3} dx}{\int_{\Gamma_0\cup\Gamma_2} P_{0,m-1}^2(x) e^{-x^3} dx}, & m > 0. \end{cases} \tag{2.22}$$

Note that

$$\int_{\Gamma_0\cup\Gamma_2} P_{0,m}^2(x) e^{-x^3} dx = (\gamma_1^{(2)} \gamma_2^{(2)} \cdots \gamma_m^{(2)})^2 \int_{\Gamma_0\cup\Gamma_2} e^{-x^3} dx, \quad m > 0, \tag{2.23}$$

and straightforward calculations give us

$$\int_{\Gamma_0\cup\Gamma_2} e^{-x^3}dx = \frac{\Gamma(1/3)}{3}(1-\omega^2), \tag{2.24}$$

$$\begin{aligned}\int_{\Gamma_0\cup\Gamma_2} P_{1,0}(x)e^{-x^3}dx &= \int_{\Gamma_0\cup\Gamma_2}(x-\beta_0^{(1)})e^{-x^3}dx\\ &= \frac{\Gamma(2/3)}{3}(1-\omega) - \frac{\Gamma(1/3)\beta_0^{(1)}}{3}(1-\omega^2)\\ &= \frac{\Gamma(2/3)}{3}(1-\omega)(1-(1+\omega)e^{\pi i/3}).\end{aligned} \tag{2.25}$$

See (1.9) for the value of $\beta_0^{(1)}$. Inserting (2.23)–(2.25) into (2.22), we arrive at

$$b_{n,n+m} = \begin{cases} \frac{n}{3\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}i, & m=0,\\ \frac{n+m}{m}(\gamma_m^{(2)})^2, & m>0, \end{cases} \tag{2.26}$$

which is (1.27) by (1.15).

We can also represent $a_{n,n+m}$ as a ratio of two integrals. Indeed, by performing similar strategies above, it is easily seen that

$$a_{n,n+m} = \frac{\int_{\Gamma_0\cup\Gamma_1} x^n P_{n,n+m}(x)e^{-x^3}dx}{\int_{\Gamma_0\cup\Gamma_1} x^{n-1}P_{n-1,n+m}(x)e^{-x^3}dx} = \frac{n}{3}\frac{\int_{\Gamma_0\cup\Gamma_1} P_{0,m}(x)e^{-x^3}dx}{\int_{\Gamma_0\cup\Gamma_1} P_{0,m+1}(x)e^{-x^3}dx}. \tag{2.27}$$

Unfortunately, this representation is not suitable for direct calculation except for $m=0$, which gives

$$\begin{aligned} a_{n,n} &= \frac{n}{3}\frac{\int_{\Gamma_0\cup\Gamma_1} P_{0,0}(x)e^{-x^3}dx}{\int_{\Gamma_0\cup\Gamma_1} P_{0,1}(x)e^{-x^3}dx}\\ &= \frac{n}{3}\frac{\int_{\Gamma_0\cup\Gamma_1} e^{-x^3}dx}{\int_{\Gamma_0\cup\Gamma_1}(x-\beta_0^{(2)})e^{-x^3}dx}\\ &= \overline{b_{n,n}} = -\frac{n}{3\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}i. \end{aligned} \tag{2.28}$$

For $m>0$ the integrals of $P_{0,m}$ over $\Gamma_0\cup\Gamma_1$ are involving polynomials orthogonal on the contour $\Gamma_0\cup\Gamma_2$, hence it is then difficult to deal with them. So we proceed in another way and we calculate the sum $a_{n,n+m}+b_{n,n+m}$. Recall the notation $\delta_{k,l}$ and $\varepsilon_{k,l}$ in (2.9). By comparing the coefficient of x^{2n+m-1} on both sides of (1.22), it follows from (2.10) that

$$a_{n,n+m}+b_{n,n+m} = \varepsilon_{n,n+m}-\varepsilon_{n+1,n+m}-c_{n,n+m}\delta_{n,n+m} = \varepsilon_{0,m}-\varepsilon_{0,m-1}-c_{n,n+m}\delta_{0,m} \tag{2.29}$$

for $m > 0$. From (2.16), we have

$$\varepsilon_{0,m} = \varepsilon_{0,m+1} + \beta_m^{(2)}\delta_{0,m} + (\gamma_m^{(2)})^2. \tag{2.30}$$

This, together with (1.15), (1.24) and (2.29), implies

$$\begin{aligned} a_{n,n+m} + b_{n,n+m} &= \varepsilon_{0,m} - \varepsilon_{0,m-1} - c_{n,n+m}\delta_{0,m} \\ &= -\beta_{m-1}^{(2)}\delta_{0,m-1} - (\gamma_{m-1}^{(2)})^2 + \beta_{m-1}^{(2)}\delta_{0,m} \\ &= \beta_{m-1}^{(2)}(\delta_{0,m} - \delta_{0,m-1}) - (\gamma_{m-1}^{(2)})^2 \\ &= -(\beta_{m-1}^{(2)})^2 - (\gamma_{m-1}^{(2)})^2 \\ &= (\gamma_m^{(2)})^2, \qquad m > 0, \end{aligned} \tag{2.31}$$

where we have made use of (2.18) in the fourth equality and the string equation (1.4) in the last step. A combination of (2.26), (2.31) and (2.28) finally gives

$$a_{n,n+m} = \begin{cases} -\frac{n}{3\sqrt{3}}\frac{\Gamma(1/3)}{\Gamma(2/3)}i, & m = 0, \\ -\frac{n}{m}(\gamma_m^{(2)})^2, & m > 0, \end{cases} \tag{2.32}$$

which is (1.26), on account of (1.15).

3 Asymptotics of a_n and b_n

From Theorem 1.3, it is clear that the coefficients in the nearest-neighbor recurrence relations are determined by a_n and b_n generated from (1.12)–(1.14). It is then interesting to study their large n behavior. In Figure 2 we have plotted the values of $a_n/n^{2/3}$ and $b_n/n^{1/3}$ for n from 0 to 70, from which we see that a_{n+1} and b_n are all strictly positive for $n \in \mathbb{N}$. We actually have the following conjecture concerning this observation.

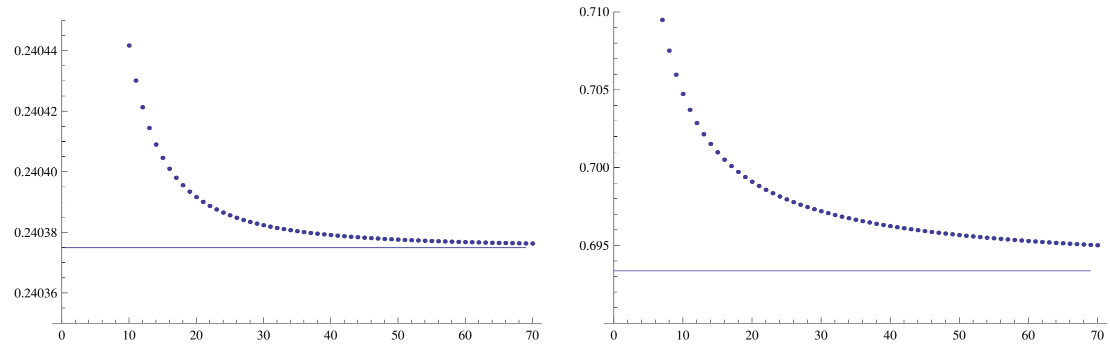


Figure 2: The values of $a_n/n^{2/3}$ (left) and $b_n/n^{1/3}$ (right) for n from 0 to 70.

Conjecture 3.1. *There is a unique positive solution of the recurrence relations* (1.12)–(1.13) *with* $a_0 = 0$ *and* $a_{n+1} > 0, b_n > 0$ *for* $n \in \mathbb{N}$. *This solution corresponds to the initial condition* $b_0 = \Gamma(2/3)/\Gamma(1/3)$.

The numerical study further suggests that the limits of $a_n/n^{2/3}$ and $b_n/n^{1/3}$ exist as $n \to \infty$, which we can identify in the proposition below.

Proposition 3.1. *Every positive solution of* (1.12)–(1.13) *has the property that*

$$\lim_{n\to\infty} a_n/n^{2/3} = \frac{1}{2\cdot 3^{2/3}}, \qquad \lim_{n\to\infty} b_n/n^{1/3} = \frac{1}{3^{1/3}}.$$

Proof. This can be proved by an argument which was already used by Freud in [12, §3]. First we show that $(a_n/n^{2/3})_{n\in\mathbb{N}}$ is a bounded sequence. From (1.12) and the positivity of a_{n+1} we find that $a_n \le b_n^2$. From (1.13) and the positivity of b_{n-1} we find $3a_nb_n \le n$ and thus also $9a_n^2b_n^2 \le n^2$. Together this gives $9a_n^3 \le n^2$, so that $0 \le a_n/n^{2/3} \le 1/9^{1/3}$.

Let $a = \liminf_{n\to\infty} a_n/n^{2/3}$ and $A = \limsup_{n\to\infty} a_n/n^{2/3}$, then $0 \le a \le A < \infty$. From (1.12) and the positivity of b_n we find $b_n = \sqrt{a_n + a_{n+1}}$. Insert this in (1.13) to find

$$3a_n\left(\sqrt{a_n+a_{n+1}} + \sqrt{a_n+a_{n-1}}\right) = n. \tag{3.1}$$

Let $n \to \infty$ in (3.1) through a subsequence for which $a_n/n^{2/3} \to a$, then one finds $1 \le 6a\sqrt{a+A}$. If $n\to\infty$ through a subsequence for which $a_n/n^{2/3} \to A$, then $6A\sqrt{a+A} \le 1$. Together this gives $6A\sqrt{a+A} \le 6a\sqrt{a+A}$. If $a+A=0$ then one automatically has $a=A=0$ so that the limit exists (further on we will see that $a \ne 0$ so that this case does not happen). If $a+A>0$ then one finds $A \le a$, and together with $a \le A$ we see that also in this case $a = A$ and the limit exists. From (1.12) we then find that $\lim_{n\to\infty} b_n/n^{1/3} = \sqrt{2a}$. If we use that information in (1.13), then $6a\sqrt{2a} = 1$, so that $a = (1/6)^{2/3} = 1/(2\cdot 3^{2/3})$. The limit for $b_n/n^{1/3}$ follows immediately from this. □

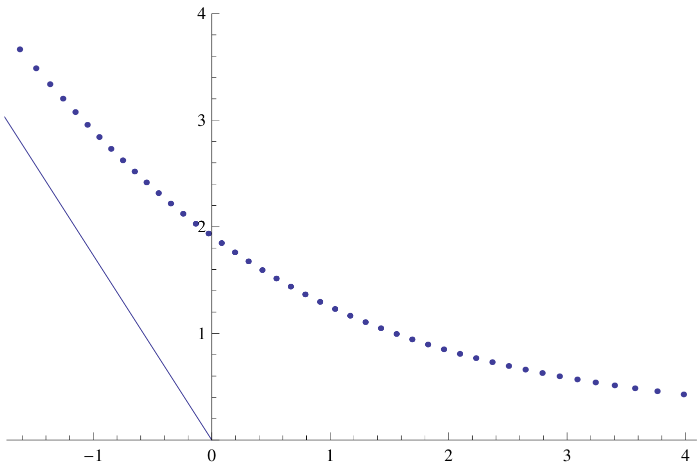


Figure 3: Zeros of $P_{45,0}$ (after scaling)

4 Zeros

The formulas in Theorems 1.2 and 1.3 can be used to generate the multiple orthogonal polynomials $P_{k,l}$ defined in (1.17) and (1.18). We investigate the distribution of their zeros numerically. If one of k and l is zero, the polynomials are orthogonal for the exponential

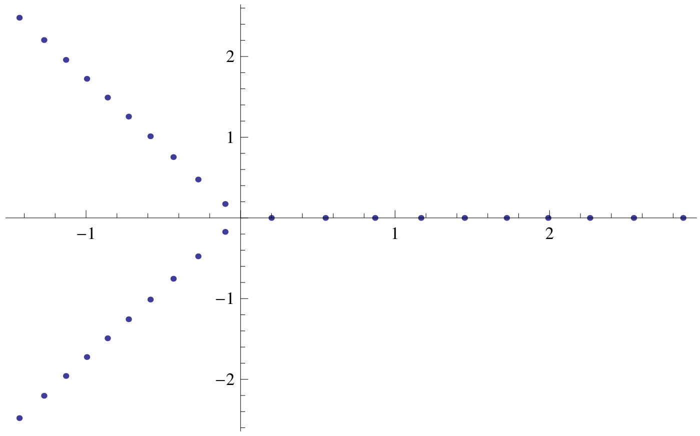


Figure 4: Zeros of $P_{15,15}$ (after scaling)

cubic weight on the curve ($\Gamma_0 \cup \Gamma_1$ or $\Gamma_0 \cup \Gamma_2$) in the complex plane. The zeros of $P_{45,0}(45^{1/3}x)$ are plotted in Figure 3. It is known that, in this case, the zeros of the polynomials, after proper scaling, will accumulate on an analytic contour in the complex plane that possesses the so-called $S-$property; cf. [14, 25]. The zero distribution was investigated earlier by Deaño, Huybrechs and Kuijlaars [8], who in fact used the weight e^{ix^3}. However, a simple rotation $x = ye^{\pi i/6}$ is enough to transform their results to the exponential cubic e^{-y^3} which we are using.

Suppose that $k = l = n$. It follows from Theorem 1.2 that

$$P_{n,n}(x) = \frac{(-1)^n}{3^n} e^{x^3} \frac{d^n}{dx^n} \left(e^{-x^3} \right). \tag{4.1}$$

We can describe the asymptotic distribution of the zeros of the diagonal polynomials $P_{n,n}$ in more detail. The main reason is that the zeros of $P_{n,n}$ are all located on the three rays $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, which simplifies matters considerably (see Figure 4). We have the following result for the diagonal polynomials. Observe that this result is the solution of Problem 59 [24, Part V, Chapter 1] for the polynomial R_n and $q = 3$.

Proposition 4.1. *The polynomials $P_{n,n}(x)$ satisfy the symmetry property $P_{n,n}(\omega x) = \omega^{2n} P_{n,n}(x)$, where $\omega = e^{2\pi i/3}$ is the primitive third root of unity. In particular*

$$P_{n,n}(x) = \begin{cases} \sum_{j=0}^{2n/3} a_j x^{3j}, & n \equiv 0 \bmod 3, \\ x^2 \sum_{j=0}^{2(n-1)/3} b_j x^{3j}, & n \equiv 1 \bmod 3, \\ x \sum_{j=0}^{2(n-2)/3+1} c_j x^{3j}, & n \equiv 2 \bmod 3, \end{cases} \tag{4.2}$$

where $(a_j)_j, (b_j)_j, (c_j)_j$ are real sequences depending on n. Furthermore the number of strictly positive real zeros of $P_{n,n}$ is

$$\begin{cases} \frac{2n}{3}, & \textit{if } n \equiv 0 \bmod 3, \\ \frac{2(n-1)}{3}, & \textit{if } n \equiv 1 \bmod 3, \\ \frac{2(n-2)}{3} + 1, & \textit{if } n \equiv 2 \bmod 3, \end{cases}$$

and $P_{n,n}(x)$ has a zero of multiplicity one at $x = 0$ when $n \equiv 2 \bmod 3$ and a zero of multiplicity two at $x = 0$ when $n \equiv 1 \bmod 3$.

Proof. We use induction on n. The symmetry property follows easily from the Rodrigues formula, so we only need to prove the result about the positive real zeros. Observe that

$$P_{0,0}(x) = 1, \quad P_{1,1}(x) = x^2, \quad P_{2,2}(x) = x(x^3 - 2/3),$$

so that the result is true for $n = 0, 1, 2$. Suppose that the result is true for $n - 1$ and let $x_1 > x_2 > \cdots > x_k > 0$ be the positive real zeros of $P_{n-1,n-1}$. Clearly the sign of $P'_{n-1,n-1}(x_j)$ is $(-1)^{j+1}$ for $1 \leq j \leq k$, hence from

$$P_{n,n}(x) = x^2 P_{n-1,n-1}(x) - \frac{1}{3} P'_{n-1,n-1}(x) \tag{4.3}$$

we find that the sign of $P_{n,n}(x_j)$ is $(-1)^j$, hence $P_{n,n}$ changes sign k times and Rolle's theorem guarantees that there are at least k zeros $y_1 > y_2 > \cdots > y_k$ with $x_j < y_j < x_{j-1}$, where $x_0 = +\infty$.

- If $n \equiv 0 \bmod 3$ then $n - 1 \equiv 2 \bmod 3$ and the induction hypothesis says that $k = 2(n-3)/3 + 1$ is odd and $P_{n-1,n-1}(x)$ has a zero of multiplicity one at $x = 0$. The sign of $P'_{n-1,n-1}(0)$ is $(-1)^k = -1$ so that the sign of $P_{n,n}(0)$ is $(-1)^{k+1} = 1$, hence there is also a zero y_{k+1} of $P_{n,n}$ between 0 and x_k, giving a total of $k+1 = 2n/3$ positive real zeros. The ω-symmetry gives another $2n/3$ zeros on Γ_1 and $2n/3$ zeros on Γ_2, which is a total of $2n$ zeros. Hence there are no other zeros of $P_{n,n}$.

- If $n \equiv 1 \bmod 3$ then $n - 1 \equiv 0 \bmod 3$ and the induction hypothesis gives $k = 2(n-1)/3$ and $P_{n-1,n-1}(x)$ has no zero at $x = 0$. Hence there will not be an additional zero between 0 and x_k so that there are $k = 2(n-1)/3$ positive real zeros for $P_{n,n}$. There is double zero of $P_{n,n}(x)$ at $x = 0$. The ω-symmetry gives another $2(n-1)/3$ zeros on Γ_1 and $2(n-1)/3$ zeros on Γ_2, hence the total number of zeros is $2(n-2) + 2 = 2n$ so that there are no other zeros.

- If $n \equiv 2 \bmod 3$ then $n - 1 \equiv 1 \bmod 3$ and the induction hypothesis gives $k = 2(n-2)/3$ is even and a double zero for $P_{n-1,n-1}(x)$ at $x = 0$. Then (4.3) implies that $P_{n,n}(x)$ has a single zero at 0. The polynomial $P_{n-1,n-1}(x)/x^2$ of degree $2n-4$ has k positive zeros and the sign of this polynomial as $x \to 0$ is $(-1)^k = 1$, so that $P_{n,n}(x)/x$ has sign $(-1)^{k+1} = -1$ as $x \to 0$. Hence $P_{n,n}(x)/x$ has a zero y_{k+1} between 0 and x_k, giving a total of $k+1 = 2(n-2)/3+1$ positive real zeros. The ω-symmetry gives another $2(n-2)/3+1$ zeros on Γ_1 and another $2(n-2)/3+1$ zeros on Γ_2, hence together with the single zero at $x = 0$ this gives a total of $2(n-2)+4 = 2n$ zeros for $P_{n,n}$ so that there are no other zeros.

□

Observe that the proof also shows that the zeros of $P_{n-1,n-1}$ and $P_{n,n}$ interlace in the sense that $x_1 < y_1 < \infty$, $x_j < y_j < x_{j-1}$ for $j = 2, \ldots, k$, $x_1 < y_1 < \infty$ and $0 < y_{k+1} < x_k$ (the latter only when $n \equiv 0 \bmod 3$ and $n \equiv 2 \bmod 3$).

We can now prove the following results

Theorem 4.2. *Let K be a compact set in $\mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$, then*

$$\lim_{n\to\infty} \frac{1}{n^{2/3}} \frac{P_{n,n}(n^{1/3}x)}{P_{n-1,n-1}(n^{1/3}x)} = \Phi(x),$$

holds uniformly for $x \in K$, where

$$\Phi(x) = \frac{1}{e^{2\pi i/3}\left(\frac{-3+\sqrt{9-4x^3}}{2}\right)^{2/3} + e^{-2\pi i/3}\left(\frac{-3-\sqrt{9-4x^3}}{2}\right)^{2/3} + 2x}.$$

Furthermore

$$\lim_{n\to\infty} \frac{1}{n^{2/3}} \frac{P'_{n,n}(n^{1/3}x)}{P_{n,n}(n^{1/3}x)} = 3x^2 - 3\Phi(x),$$

holds uniformly for $x \in K$.

Proof. Consider the ratio

$$\frac{1}{N}\frac{\frac{d}{dx}P_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)} = \frac{1}{N}\sum_{j=1}^{2n}\frac{1}{x - x_{j,n}/N^{1/3}},$$

where $\{x_{j,n}, 1 \le j \le 2n\}$ are the zeros of $P_{n,n}$ which are all on the set $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, then if $x \in K$ we have

$$\left|\frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)}\right| \le \frac{1}{N}\sum_{j=1}^{2n}\frac{1}{|x - x_{j,n}/N^{1/3}|} \le \frac{2n}{N\delta},$$

where $\delta = \inf\{|x-y| \;:\; x \in K, y \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2\} > 0$ is the minimal distance between K and $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. If $n/N \to 1$ we then see that the family of analytic functions

$$\frac{1}{N^{2/3}}\frac{P'_{n,n}(N^{1/3}x)}{P_{n,n}(N^{1/3}x)},$$

is uniformly bounded on K. By Montel's theorem there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}}\frac{P'_{n_k-2,n_k-2}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)} = F(x), \tag{4.4}$$

uniformly for $x \in K$, where F is an analytic function on K for which $F(x) = 2/x + \mathcal{O}(1/x^2)$ as $x \to \infty$. This function F may depend on the selected subsequence, so our aim is to prove that it is independent of the subsequence.

Now consider (4.3) for P_{n_k-1,n_k-1}, then

$$\frac{1}{N^{2/3}}\frac{P_{n_k-1,n_k-1}(N^{1/3}x)}{P_{n_k-2,n_k-2}(N^{1/3}x)} = x^2 - \frac{1}{3N^{2/3}}\frac{P'_{n_k-2,n_k-2}(N^{1/3}x)}{P_{n_k-2,n_k-2}(N^{1/3}x)},$$

hence (4.4) implies (with $N = n_k$)

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}}\frac{P_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)} = \Phi(x), \tag{4.5}$$

uniformly on K, where $\Phi(x) = x^2 - F(x)/3$. This uniform convergence of analytic functions implies also the uniform convergence of the derivatives, hence

$$\begin{aligned}
\Phi'(x) &= \lim_{n_k\to\infty}\left(\frac{1}{n_k^{2/3}}\frac{P_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)}\right)' \\
&= \lim_{n_k\to\infty}\frac{1}{n_k^{2/3}}\frac{P_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)}\left(\frac{n_k^{1/3}P'_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} - \frac{n_k^{1/3}P'_{n_k-2,n_k-2}(n_k^{1/3}x)}{P_{n_k-2,n_k-2}(n_k^{1/3}x)}\right),
\end{aligned}$$

but this means that

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}} \frac{P'_{n_k-1,n_k-1}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} = F(x), \tag{4.6}$$

uniformly on K, with the same limit as in (4.4). But then (4.3) implies that

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k,n_k}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} = \Phi(x), \tag{4.7}$$

uniformly on K, with the same limit as in (4.5). We can repeat this reasoning once more and find that

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}} \frac{P'_{n_k,n_k}(n_k^{1/3}x)}{P_{n_k,n_k}(n_k^{1/3}x)} = F(x), \tag{4.8}$$

and

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k+1,n_k+1}(n_k^{1/3}x)}{P_{n_k,n_k}(n_k^{1/3}x)} = \Phi(x), \tag{4.9}$$

uniformly on K.

We will show that the function Φ satisfies a cubic equation, from which we can determine Φ and hence also F uniquely, so that Φ and F do not depend on the selected subsequence $(n_k)_{k\in\mathbb{N}}$. Consider the nearest neighbor recurrence relations for the diagonal $n=m$

$$\begin{aligned} xP_{n,n}(x) &= P_{n+1,n}(x) + c_{n,n}P_{n,n}(x) + a_{n,n}P_{n-1,n}(x) + b_{n,n}P_{n,n-1}(x) && (4.10)\\ xP_{n,n}(x) &= P_{n,n+1}(x) + d_{n,n}P_{n,n}(x) + a_{n,n}P_{n-1,n}(x) + b_{n,n}P_{n,n-1}(x). && (4.11) \end{aligned}$$

Subtracting (4.10) and (4.11) gives

$$P_{n+1,n}(x) - P_{n,n+1}(x) = (d_{n,n} - c_{n,n})P_{n,n}(x).$$

Use this for $n \to n-1$ to eliminate $P_{n-1,n}(x)$ in (4.10) to find

$$\begin{aligned} xP_{n,n}(x) = P_{n+1,n}(x) + c_{n,n}P_{n,n}(x) + (a_{n,n} + b_{n,n})P_{n,n-1}(x) \\ + \ a_{n,n}(c_{n-1,n-1} - d_{n-1,n-1})P_{n-1,n-1}(x). \end{aligned}$$

From Theorem 1.3 we have

$$c_{n,n} = \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3}, \quad d_{n,n} = \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{-\pi i/3},$$

so that $c_{n-1,n-1} - d_{n-1,n-1} = i\sqrt{3}\Gamma(2/3)/\Gamma(1/3)$. Furthermore

$$a_{n,n} = -\frac{ni}{3\sqrt{3}} \frac{\Gamma(1/3)}{\Gamma(2/3)}, \quad b_{n,n} = \frac{ni}{3\sqrt{3}} \frac{\Gamma(1/3)}{\Gamma(2/3)},$$

so that the recurrence relation becomes

$$xP_{n,n}(x) = P_{n+1,n}(x) + \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3} P_{n,n}(x) + \frac{n}{3} P_{n-1,n-1}(x). \tag{4.12}$$

Use this for $n^{1/3}x$ and divide by $P_{n,n}(n^{1/3}x)$, then

$$x = \frac{1}{n^{1/3}} \frac{P_{n+1,n}(n^{1/3}x)}{P_{n,n}(n^{1/3}x)} + \frac{1}{n^{1/3}} \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3} + \frac{n}{3n^{1/3}} \frac{P_{n-1,n-1}(n^{1/3}x)}{P_{n,n}(n^{1/3}x)},$$

and by using (4.7) we find

$$\lim_{n_k\to\infty} \frac{1}{n_k^{1/3}} \frac{P_{n_k+1,n_k}(n_k^{1/3}x)}{P_{n_k,n_k}(n_k^{1/3}x)} = x - \frac{1}{3\Phi(x)}, \tag{4.13}$$

uniformly on K. We can repeat the reasoning for $n \to n-1$ and use (4.5) to find

$$\lim_{n_k\to\infty} \frac{1}{n_k^{1/3}} \frac{P_{n_k,n_k-1}(n_k^{1/3}x)}{P_{n_k-1,n_k-1}(n_k^{1/3}x)} = x - \frac{1}{3\Phi(x)}, \tag{4.14}$$

uniformly on K. But then the uniform convergence also holds for the derivative, and as before (4.6) then implies that

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}} \frac{P'_{n_k,n_k-1}(n_k^{1/3}x)}{P_{n_k,n_k-1}(n_k^{1/3}x)} = F(x).$$

Use (4.3) for $P_{n+1,n}(n^{1/3}x)$ and divide by $P_{n,n-1}(n^{1/3}x)$, then the latter asymptotic result gives

$$\lim_{n_k\to\infty} \frac{1}{n_k^{2/3}} \frac{P_{n_k+1,n_k}(n_k^{1/3}x)}{P_{n_k,n_k-1}(n_k^{1/3}x)} = x^2 - \frac{1}{3}F(x) = \Phi(x), \tag{4.15}$$

uniformly on K.

In a similar way as before, the nearest neighbor recurrence relations for $(n+1,n)$ can be transformed to

$$\begin{aligned} xP_{n+1,n}(x) = P_{n+1,n+1}(x) + d_{n+1,n}P_{n+1,n}(x) + (a_{n+1,n} + b_{n+1,n})P_{n,n}(x) \\ +\ b_{n+1,n}(d_{n,n-1} - c_{n,n-1})P_{n,n-1}(x). \end{aligned}$$

From Theorem 1.3 we now use

$$d_{n+1,n} = -b_0 e^{\pi i/3}, \quad c_{n+1,n} = b_1 e^{\pi i/3},$$

so that $d_{n,n-1} - c_{n,n-1} = -(b_0 + b_1)e^{\pi i/3} = -e^{\pi i/3}/(3a_1)$, where we used (1.13) with $n = 1$. We also have

$$a_{n+1,n} = (n+1)a_1 e^{-\pi i/3}, \quad b_{n+1,n} = -na_1 e^{-\pi i/3},$$

so that the recurrence relation becomes

$$xP_{n+1,n}(x) = P_{n+1,n+1}(x) - \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3} P_{n+1,n}(x) + a_1 e^{-\pi i/3} P_{n,n}(x) + \frac{n}{3} P_{n,n-1}(x). \tag{4.16}$$

Consider this for $n^{1/3}x$ and divide by $P_{n+1,n}(n^{1/3}x)$ then

$$\begin{aligned} x = \frac{1}{n^{1/3}} \frac{P_{n+1,n+1}(n^{1/3}x)}{P_{n+1,n}(n^{1/3}x)} - \frac{1}{n^{1/3}} \frac{\Gamma(2/3)}{\Gamma(1/3)} e^{\pi i/3} + \frac{a_1}{n^{1/3}} e^{-\pi i/3} \frac{P_{n,n}(n^{1/3}x)}{P_{n+1,n}(n^{1/3}x)} \\ + \frac{n}{3n^{1/3}} \frac{P_{n,n-1}(n^{1/3}x)}{P_{n+1,n}(n^{1/3}x)} \end{aligned}$$

and by using (4.13) and (4.15) we find

$$\lim_{n_k\to\infty} \frac{1}{n_k^{1/3}} \frac{P_{n_k+1,n_k+1}(n_k^{1/3}x)}{P_{n_k+1,n_k}(n_k^{1/3}x)} = x - \frac{1}{3\Phi(x)}, \tag{4.17}$$

uniformly on K.

Now use the relation

$$\frac{1}{n^{2/3}} \frac{P_{n+1,n+1}(n^{1/3}x)}{P_{n,n}(n^{1/3}x)} = \frac{1}{n^{1/3}} \frac{P_{n+1,n+1}(n^{1/3}x)}{P_{n+1,n}(n^{1/3}x)} \frac{1}{n^{1/3}} \frac{P_{n+1,n}(n^{1/3}x)}{P_{n,n}(n^{1/3}x)}$$

and let $n \to \infty$ through the subsequence $(n_k)_{k\in\mathbb{N}}$, then (4.9), (4.13) and (4.17) show that

$$\Phi(x) = \left(x - \frac{1}{3\Phi(x)} \right)^2. \tag{4.18}$$

The cubic equation (4.18) has one solution Φ_1 which behaves for $x \to \infty$ as

$$\Phi_1(x) = x^2 + \mathcal{O}(1/x), \qquad x \to \infty.$$

There are two other solutions $\Phi_{2,3}$ which behave as $1/(3x)$ as $x \to \infty$

$$\Phi_2(x) = \frac{1}{3x} + \frac{1}{\sqrt{27}x^{5/2}} + \mathcal{O}(1/x^4), \quad \Phi_3(x) = \frac{1}{3x} - \frac{1}{\sqrt{27}x^{5/2}} + \mathcal{O}(1/x^4), \qquad x \to \infty.$$

Recall that our Φ satisfies $\Phi(x) = x^2 - F(x)/3$, where $F(x) = \mathcal{O}(1/x)$, so that we need the solution Φ_1. The discriminant of (4.18) is $(4x^3 - 9)/27$ so that Φ_1 has branch points at $(9/4)^{1/3}$, $(9/4)^{1/3}e^{\pm 2\pi i/3}$, which are three points on Γ_0, Γ_1, Γ_2 respectively, and since all the zeros of $P_{n,n}$ are on $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, we conclude that the scaled zeros $x_{j,n}/n^{1/3}$ are dense on the three segments $[0, (9/4)^{1/3}] \cup [0, (9/4)^{1/3}e^{2\pi i/3}] \cup [0, (9/4)^{1/3}e^{-2\pi i/3}]$.

The cubic equation can be solved explicitly by using Cardano's formula: let $y = x - 1/(3\Phi)$ and $z = 1/y$, then the cubic equation (4.18) becomes

$$z^3 - 3xz + 3 = 0,$$

and the solutions are $z = \omega^j u^{1/3} + \omega^{-j} v^{1/3}$ ($j = 0, 1, 2$), where $\omega = e^{2\pi i/3}$, $u + v = -3$ and $uv = x^3$, i.e.,

$$u = \frac{-3 + \sqrt{9 - 4x^3}}{2}, \quad v = \frac{-3 - \sqrt{9 - 4x^3}}{2} = \frac{2x^3}{-3 + \sqrt{9 - 4x^3}}.$$

The solution Φ_1 corresponds to the solution with $z(x) = 1/x + \mathcal{O}(1/x^4)$, and this is $z(x) = \omega^2 u^{1/3} + \omega^{-2} v^{1/3}$, and since $\Phi = y^2$, we find

$$\Phi_1(x) = \frac{1}{(\omega^2 u^{1/3} + \omega^{-2} v^{1/3})^2} = \frac{1}{\omega u^{2/3} + \omega^{-1} v^{2/3} + 2x}.$$

$\square$

Corollary 4.3. *Let $\{x_{j,n}, j = 1, 2, \dots, 2n\}$ be the zeros of $P_{n,n}$ and μ_n be the normalized counting measure of the scaled zeros $x_{j,n}/n^{1/3}$,*

$$\mu_n = \frac{1}{2n} \sum_{j=1}^{2n} \delta_{x_{j,n}/n^{1/3}}.$$

Then the sequence $(\mu_n)_n$ converges weakly to the probability measure μ for which

$$\int f(x)\, d\mu(x) = \int_0^{(9/4)^{1/3}} v(x) f(x)\, dx + \int_0^{\omega(9/4)^{1/3}} v(x) f(x)\, dx + \int_0^{\omega^2(9/4)^{1/3}} v(x) f(x)\, dx,$$

where $\omega = e^{2\pi i/3}$ and

$$v(x) = \frac{\sqrt{3}}{4\pi} \Big(1 + x[a(x) + b(x)]\Big) [b(x) - a(x)], \tag{4.19}$$

with

$$a(x) = \left(\frac{3 - \sqrt{9 - 4x^3}}{2}\right)^{1/3}, \quad b(x) = \left(\frac{3 + \sqrt{9 - 4x^3}}{2}\right)^{1/3}. \tag{4.20}$$

Proof. The Stieltjes transform of the measure μ_n is

$$\int \frac{1}{x - t}\, d\mu_n(t) = \frac{1}{2n^{2/3}} \frac{P'_{n,n}(n^{1/3}x)}{P_{n,n}(n^{1/3}x)},$$

hence Theorem 4.2 gives

$$\lim_{n\to\infty} \int \frac{1}{x - t}\, d\mu_n(t) = \frac{1}{2} F(x) = \frac{3}{2}\Big(x^2 - \Phi(x)\Big),$$

uniformly on compact sets of $\mathbb{C} \setminus (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$. The Grommer-Hamburger theorem [13] then implies that μ_n converges weakly to a measure μ for which

$$\int \frac{1}{x - t}\, d\mu(t) = \frac{3}{2}\Big(x^2 - \Phi(x)\Big).$$

The function Φ is analytic in $\mathbb{C} \setminus ([0, (9/4)^{1/3}] \cup [0, \omega(9/4)^{1/3}] \cup [0, \omega^2(9/4)^{1/3}])$, hence the measure μ is supported on $[0, (9/4)^{1/3}] \cup [0, \omega(9/4)^{1/3}] \cup [0, \omega^2(9/4)^{1/3}]$. Furthermore it is absolutely continuous and we can find the density by using the Stieltjes inversion formula

$$v(x) = -\frac{1}{\pi} \lim_{\epsilon\to 0+} \Im \frac{3}{2}\Big((x + i\epsilon)^2 - \Phi(x + i\epsilon)\Big).$$

Due to the ω-symmetry, it is sufficient to determine $v(x)$ for $x \in [0, (9/4)^{1/3}]$. Clearly

$$v(x) = \frac{3}{2\pi} \lim_{\epsilon\to 0+} \Im \Phi(x + i\epsilon) = \frac{3}{2\pi} \Im \frac{1}{(\omega^2 a + \omega^{-2} b)^2},$$

with a and b given in (4.20). Then by some elementary (complex) calculus, using $(a^2 - ab + b^2)(a + b) = a^3 + b^3$ and $ab = x$, one finds the expression (4.19) for the density v. ☐

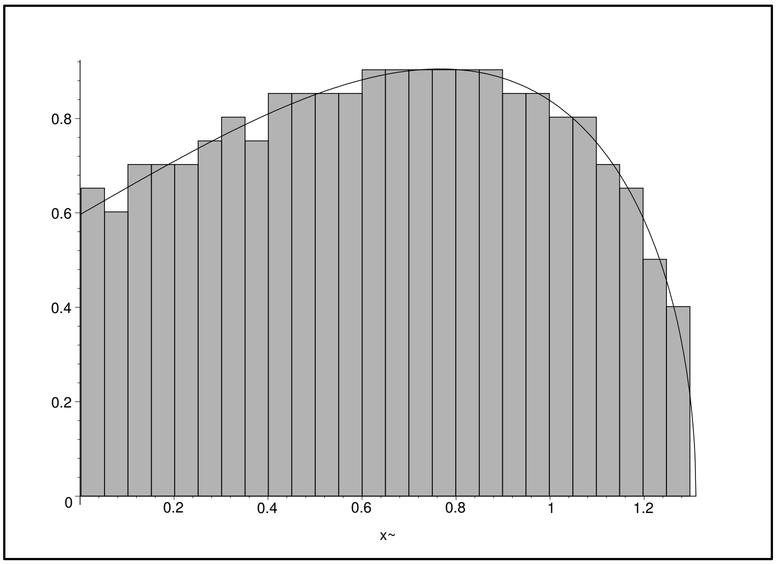


Figure 5: Histogram of the real zeros of $P_{600,600}$ and the density $3v(x)$ on $[0,(9/4)^{1/3}]$

In Figure 5 we have given a histogram of the 400 real zeros of $P_{600,600}$ together with the density v, scaled so as to have total mass one for all the real zeros. There are 400 zeros on the interval $[0,\omega(9/4)^{1/3}]$ and 400 zeros on the interval $[0,\omega^2(9/4)^{1/3}]$ and these zeros are obtained by rotating the real zeros over an angle $\pm 2\pi/3$. The density v has a finite non-zero value at the origin $v(0) = 3^{1/3}\sqrt{3}/4\pi = 0.198788$ and tends to zero as $\sqrt{(9/4)^{1/3} - x}$ when $x \to (9/4)^{1/3}$.

If $k \neq l$, the rotational symmetry of the zeros is broken. Suppose that $l \geq 2k$, then we see numerically that k zeros of $P_{k,l}$ lie on the line containing Γ_1 (some zeros are in fact on $-\Gamma_1$), while the other zeros are distributed on a complex contour in the lower half plane; see Figure 6. Similarly, if $k \geq 2l$, then l zeros of $P_{k,l}$ lie on the line containing Γ_2 (again some zeros are on $-\Gamma_2$), and the other zeros are distributed on a complex contour in the upper half plane, as illustrated in Figure 7. Indeed, from Theorem 1.2, it is easily seen that the zeros of $P_{k,l}$ are complex conjugates of the zeros of $P_{l,k}$. We expect that the asymptotic zero distribution of $P_{k,l}$ will depend on the limit of the ratio k/l.

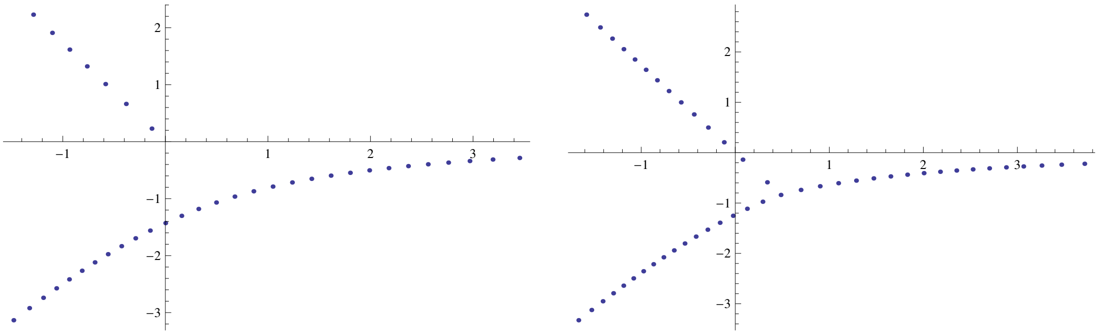
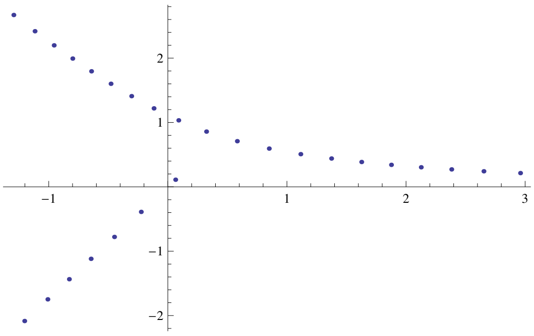


Figure 6: Zeros of $P_{7,30}$ (left) and $P_{14,35}$ (right) after scaling



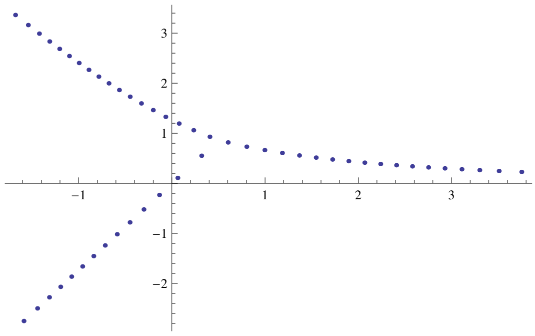


Figure 7: Zeros of $P_{20,7}$ (left) and $P_{36,14}$ (right) after scaling

5 Conclusions and outlook

In this paper, we have introduced the multiple orthogonal polynomials associated with an exponential cubic weight e^{-x^3} over two contours in the complex plane. The basic properties of these polynomials are studied, which include the Rodrigues formula and nearest-neighbor recurrence relations. These results then allow us to perform numerical studies of the recurrence coefficients and zero distributions of the multiple orthogonal polynomials. Moreover, the recurrence coefficients are related to a discrete Painlevé equation. One can also consider the more general exponential cubic weight e^{-x^3+tx}, where $t \in \mathbb{R}$, and the associated multiple orthogonal polynomials have similar Rodrigues formulas and nearest-neighbor recurrence relations. Indeed, with e^{-x^3} replaced by e^{-x^3+tx} in Theorem 1.2, the difference-differential equations (2.6) and (2.11) now read

$$P_{n,n+m}(x) = (x^2 - \frac{t}{3})P_{n-1,n+m-1}(x) - \frac{1}{3}P'_{n-1,n+m-1}(x),$$
$$P_{n+m,n}(x) = (x^2 - \frac{t}{3})P_{n+m-1,n-1}(x) - \frac{1}{3}P'_{n+m-1,n-1}(x).$$

This then implies that

$$\delta_{n,n+m} = \delta_{0,m}, \qquad \varepsilon_{n,n+m} = \varepsilon_{0,m} - \frac{t}{3},$$
$$\delta_{n+m,n} = \delta_{m,0}, \qquad \varepsilon_{n+m,n} = \varepsilon_{m,0} - \frac{t}{3},$$

where $\delta_{k,l}$ and $\varepsilon_{k,l}$ are defined in (2.9) and now depend on t. Following the same strategy as in the proof of Theorem 1.3 and using the string equation (A.7) at the final stage, we obtain (1.22) and (1.23) with

$$c_{n,n+m} = \begin{cases} \beta_0^{(1)}(t), & m = 0, \\ -\beta_{m-1}^{(2)}(t), & m > 0, \end{cases}$$
$$d_{n,n+m} = \beta_m^{(2)}(t),$$
$$a_{n,n+m} = \begin{cases} \frac{n}{3\sqrt{3}} \frac{1}{\beta_0^{(1)}(t) - \beta_0^{(2)}(t)}, & m = 0, \\ -\frac{n}{m}(\gamma_m^{(2)}(t))^2 - \frac{t}{3}, & m > 0, \end{cases}$$
$$b_{n,n+m} = \begin{cases} \frac{n}{3} \frac{1}{\beta_0^{(2)}(t) - \beta_0^{(1)}(t)}, & m = 0, \\ \frac{(n+m)}{m}(\gamma_m^{(2)}(t))^2, & m > 0, \end{cases}$$

where $\beta_n^{(1)}(t), (\gamma_n^{(1)})^2(t)$ are the recurrence coefficients of the monic orthogonal polynomials with respect to the weight e^{-x^3+tx} on $\Gamma_0 \cup \Gamma_1$, and $\beta_n^{(2)}(t), (\gamma_n^{(2)})^2(t)$ are the recurrence coefficients of the monic orthogonal polynomials with respect to the weight e^{-x^3+tx} on $\Gamma_0 \cup \Gamma_2$. The recurrence relations (1.28) and (1.29) for this general case can be obtained similarly, but we omit the results here. Clearly, in the general case, we lose the nice structure of the recurrence coefficients as stated in Theorem 1.3, and more importantly we lose the symmetry given in Proposition 4.1, which is why we focus on the weight e^{-x^3} in this paper.

The challenging problem is to establish the asymptotic zero distribution of $P_{k,l}(x)$ for the non-symmetric case. At present we are unable to find an analogue of Theorem 4.2 and Corollary 4.3 because of two reasons: first one needs the asymptotic behavior of the recurrence coefficients and at present we can only conjecture the behavior (see Proposition 3.1). If we assume this to be correct, then the proof of Theorem 4.2 can be used to find the asymptotic behavior, away from the set where the zeros of the multiple orthogonal polynomials accumulate, in terms of an algebraic function Φ satisfying a cubic equation. But the second reason is that we don't know where the zeros of the multiple orthogonal polynomials accumulate. The discriminant of the cubic equation is a quartic polynomial in x and the four roots are branch points of the algebraic function Φ. The zeros will accumulate on two curves, each connecting two points in the complex plane, see Figures 6 and 7. One of the curves is a straight line, the other is a curved line connecting two of the four branch points. The straight line, however, does not connect the other two branch points but starts from one branch point and stops before the second branch point is reached. This suggests that a vector equilibrium problem is involved, for two measures living on curves connecting four branch points, with an external field x^3 induced by the weight e^{-x^3}. In order to characterize the limiting zero distribution, one may need to extend the concept of S-property (cf. [14, 25]) for orthogonal polynomials and equilibrium measures to this setting for multiple orthogonal polynomials and vector equilibrium problems. Once that is obtained, one may be able to use the Riemann-Hilbert method to find the asymptotic behavior of the multiple orthogonal polynomials.

Appendix

A Derivation of the string equations

In this appendix, we give an alternative proof of the string equations (1.4)–(1.5) using ladder operators for orthogonal polynomials. Note that the ladder operators for multiple orthogonal polynomials and their compatibility conditions can be found in [10].

Following the general set-up (cf. [5]), if the weight function w vanishes at the endpoints of the orthogonality interval, the lowering and raising ladder operators for the associated monic polynomials p_n are given by

$$\left(\frac{d}{dx} + B_n(x)\right) p_n(x) = \gamma_n^2 A_n(x) p_{n-1}(x), \tag{A.1}$$

$$\left(\frac{d}{dx} - B_n(x) - \mathsf{v}'(x)\right) p_{n-1}(x) = -A_{n-1}(x) p_n(x), \tag{A.2}$$

with

$$\mathsf{v}(x) := -\ln w(x),$$

and

$$A_n(x) := \frac{1}{h_n}\int \frac{\mathsf{v}'(x)-\mathsf{v}'(y)}{x-y}\,[p_n(y)]^2 w(y)dy, \tag{A.3}$$

$$B_n(x) := \frac{1}{h_{n-1}}\int \frac{\mathsf{v}'(x)-\mathsf{v}'(y)}{x-y}\,p_{n-1}(y)p_n(y)w(y)dy, \tag{A.4}$$

where

$$\int p_m(x)p_n(x)\omega(x)dx = h_n\delta_{m,n}, \quad m,n=0,1,2,\dots. \tag{A.5}$$

Note that A_n and B_n are not independent, but satisfy the following compatibility conditions [18, Lemma 3.2.2 and Theorem 3.2.4].

Proposition A.1. *The functions A_n and B_n defined in (A.3) and (A.4) satisfy*

$$B_{n+1}(x)+B_n(x) = (x-\beta_n)A_n(x)-\mathsf{v}'(x), \tag{S_1}$$

$$1+(x-\beta_n)[B_{n+1}(x)-B_n(x)] = \gamma_{n+1}^2A_{n+1}(x)-\gamma_n^2A_{n-1}(x). \tag{S_2}$$

Now we consider a more general exponential cubic weight e^{-x^3+tx}, with parameter $t\in\mathbb{R}$. Then

$$\mathsf{v}(x) = -\ln w(x) = x^3-tx,$$

and

$$\frac{\mathsf{v}'(x)-\mathsf{v}'(y)}{x-y} = 3(x+y).$$

It then follows from (A.3)–(A.5) that

$$A_n(x) = 3(x+\beta_n), \qquad B_n(x) = 3\gamma_n^2. \tag{A.6}$$

Substituting (A.6) into (S_1) and comparing the coefficients of the constant term, we have

$$\gamma_n^2+\gamma_{n+1}^2+\beta_n^2-\frac{t}{3} = 0. \tag{A.7}$$

From (S_2) we similarly get

$$3\gamma_n^2(\beta_{n-1}+\beta_n) = n. \tag{A.8}$$

Note that in this case, the recurrence coefficients β_n and γ_n^2 all depend on t. By setting $t=0$ in (A.7) and (A.8), we recover the string equations (1.4) and (1.5).

The weight e^{-x^3+tx} is a modification of the weight e^{-x^3} with an exponential factor e^{tx}, and as a consequence the recurrence coefficients satisfy the Toda equations [18, §2.8]

$$\frac{d}{dt}\gamma_n^2 = \gamma_n^2(\beta_n-\beta_{n-1}), \tag{A.9}$$

$$\frac{d}{dt}\beta_n = \gamma_{n+1}^2-\gamma_n^2. \tag{A.10}$$

If we differentiate (A.10) and then use (A.9), we find

$$\beta_n''(t) = \gamma_{n+1}^2(\beta_{n+1}-\beta_n)-\gamma_n^2(\beta_n-\beta_{n-1}).$$

Then use (A.7) and (A.8) to find

$$\beta_n''(t) = 2\beta_n^3 - \frac{2t}{3}\beta_n + \frac{2n+1}{3},$$

which is the Painlevé II equation. The equations (A.7) and (A.8) give

$$\frac{n}{\beta_n + \beta_{n-1}} + \frac{n+1}{\beta_{n+1} + \beta_n} + 3\beta_n^2 = t$$

which also follows from the Bäcklund transformation of the second Painlevé equation (see [11] and [7]), hence it is not surprising to find that β_n satisfies the Painlevé II equation.

If we write $x_n(t) = a\beta_n(-at)$, where $a = (3/2)^{1/3}$, then

$$x_n''(t) = 2x_n^3 + tx_n + n + \frac{1}{2},$$

which is the Painlevé II equation in standard form and with parameter $\alpha = n + \frac{1}{2}$. The second Painlevé equation is closely related to the Airy equation and has special solutions in terms of Airy functions for the parameter values $\alpha = n + \frac{1}{2}$, with $n \in \mathbb{Z}$ [6, §7.1]. This relation with the Airy function was to be expected since one has the integral representation

$$\mathrm{Ai}(t) = \frac{1}{2\pi i} \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{z^3/3 - tz}\, dz,$$

(see Eq. 9.5.4 of the NIST Digital Library of Mathematical Functions*) which contains (a slight variation of) the weight e^{-z^3+tz}. The special solution of Painlevé II in terms of Airy functions is

$$x_n(t) = \frac{\tau_n'(t)}{\tau_n(t)} - \frac{\tau_{n+1}'(t)}{\tau_{n+1}(t)},$$

where τ_n is the Hankel matrix

$$\tau_n(t) = \begin{pmatrix} \varphi(t) & \varphi'(t) & \cdots & \varphi^{(n-1)}(t) \\ \varphi'(t) & \varphi''(t) & \cdots & \varphi^{(n)}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \varphi^{(n-1)}(t) & \varphi^{(n)}(t) & \cdots & \varphi^{(2n-2)}(t) \end{pmatrix},$$

and φ is a solution of the Airy equation $\varphi'' + \frac{1}{2}t\varphi = 0$. This solution of P_{II} coincides with the well known solution

$$\beta_n = \delta_n - \delta_{n+1},$$

where δ_n is the coefficient of x^{n-1} for the monic orthogonal polynomial $P_n(x) = x^n + \delta_n x^{n-1} + \cdots$. One has $\delta_n = \Delta_n^*/\Delta_n$, where Δ_n is the Hankel matrix containing the moments

$$\Delta_n = \begin{pmatrix} m_0 & m_1 & \cdots & m_{n-1} \\ m_1 & m_2 & \cdots & m_n \\ \vdots & \vdots & \cdots & \vdots \\ m_{n-1} & m_n & \cdots & m_{2n-2} \end{pmatrix}$$

*`http://dlmf.nist.gov`

and Δ_n^* is a similar determinant but with the last column replaced by $m_n, m_{n+1}, \ldots, m_{2n-1}$ respectively. Even though this is an explicit solution, it is not convenient for finding the recurrence coefficients when n is large because of the high number of computations involved, whereas the relations (A.7)–(A.8) have a low computational complexity. The explicit solution is also not convenient for obtaining the asymptotic behavior of β_n and a_n^2 as $n \to \infty$, which is easier to obtain from the string equations (see Proposition 3.1).

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